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Angular-Rate Estimation Using Delayed Quaternion Measurements

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Algorithms are presented for estimating the angular-rate vector of satellites using quaternion measurements. Two approaches are compared, one that uses differentiated quaternion measurements to yield coarse rate measurements, which are then fed into two different estimators. In the other approach, the raw quaternion measurements themselves are fed directly into the two estimators. The two estimators rely on the ability to decompose the nonlinear part of the rotational dynamics equation of a body into a product of an angular-rate dependent matrix and the angular-rate vector itself. This nonunique decomposition enables the treatment of the nonlinear spacecraft dynamics model as a linear one and, thus, the application of a pseudolinear Kalman filter. It also enables the application of a special Kalman filter, which is based on the use of the solution of the state–dependent algebraic Riccati equation to compute the gain matrix and, thus, eliminates the need to compute recursively the filter covariance matrix. The replacement of the rotational dynamics by a simple Markov model is also examined. Special consideration is given to the problem of delayed quaternion measurements. Two solutions to this problem are suggested and tested. Real Rossi X-Ray Timing Explorer data are used to test these algorithms, and results are presented.

I. Introduction

IN most spacecraft (SC) there is a need to know its angular rate. Precise angular rate is required for attitude determination, and a coarse rate is needed for attitude control damping. Classically, angular-rate information is obtained from gyros. These days, there is a tendency to build smaller, lighter, and cheaper SC. Many missions require accurate rate information for control purposes, for example, jitter control, in which gyros are essential. However, several other missions exist where control can be achieved without accurate gyro information (for example, the SAMPEX mission[1]). We address these missions in this paper. Therefore, the inclination now is to do away with gyros and use other means to determine the SC angular rate. The latter is also needed even in gyro-equipped satellites when the angular rate is out of range of the SC gyros.

There are several ways to obtain the angular rate in a gyroless SC. When the attitude is known, one can differentiate the attitude in whatever parameters it is given and use the kinematics equation that connects the derivative of the attitude with the satellite angular rate to compute the latter.[2] However, the differentiation of the attitude introduces a considerable noise component in the computed angular-rate vector. To overcome this noise, the computed rate components can be filtered by a passive low-pass filter. This, however, introduces a delay in the computed rate.[2] When using an active filter, such as a Kalman filter (KF), the delay can be eliminated.[3,4]

Another approach may also be adopted to the problem of angular-rate computation, where the vector measurements themselves are differentiated. This approach was used by Natanson[5] for estimating attitude from magnetometer measurements and by Challa et al.[6] to obtain attitude as well as rate. Similarly, Challa et al.[7] used derivatives of the Earth magnetic field vector to obtain attitude and rate. This approach, too, introduces noise in the computed rate.

All of these methods use the derivative of either the attitude parameters or of the measured directions, which normally determine the attitude parameters. Another approach is that of using the attitude parameters, or the measured directions themselves, as measurements in some kind of a KF. In this case, the kinematics equation that connects the attitude parameters, or the directions, with their derivatives are included in the dynamics equation used by the filter, and thereby, as will be shown, the need for differentiation is eliminated.[8,9]

New sensor packages that yield the SC attitude in terms of the attitude quaternion are now available. Therefore, it is possible to use the quaternion supplied by such sensors as measurements and, as mentioned before, eliminate the need for differentiation. In this paper we investigate this possibility.

We will apply two special KFs that make use of the SC angular dynamics model; therefore, by way of introduction, in the next section we present the development of the SC dynamics model, and in Sec. III, we present the two filters. For comparison purposes, in Sec. IV, we treat the approach where the angular rate is still extracted from derivatives, but here we pass the resultant noisy quaternion through the two active rather than through a passive filter as was done in Ref. 3. The other approach, where the raw quaternion measurements themselves are fed into the filter, requires the addition of the quaternion to the state vector, which comprises the angular-rate vector. This is treated in Sec. V. In Sec. VI, we consider the case where the filter dynamics are drastically simplified by reducing the dynamics equation of the SC to a first-order Markov process. The issue of delayed quaternion measurements is presented in Sec. VII, and the last section is the conclusions section.

II. Filter Dynamics Model

The dynamics model is that which describes the propagation of the SC angular velocity ω. The angular dynamics of a constant mass SC is given in the following equation[10]:

$$I\dot{\omega} + \dot{\boldsymbol{h}} + \omega \times (I\omega + \boldsymbol{h}) = \boldsymbol{T} \tag{1}$$

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where $\omega^T = [\omega_x, \omega_y, \omega_z]$, I is the SC inertia tensor, $\boldsymbol{h}$ is the momentum of the momentum wheels, and $\boldsymbol{T}$ is the external torque operating on the SC. The components ω_x, ω_y, and ω_z are the three components of the sought angular-rate vector ω of the SC body with respect to inertial space when resolved in the body coordinates. Equation (1) can be written as

$$\dot{\omega} = -I^{-1}[\omega \times (I\omega + \boldsymbol{h})] + I^{-1}(\boldsymbol{T} - \dot{\boldsymbol{h}}) \tag{2}$$

Denote the cross product matrix of the vector $(I\omega + \boldsymbol{h})$ by $[(I\omega + \boldsymbol{h})\times]$, and define

$$F(\omega) = I^{-1}[(I\omega + \boldsymbol{h})\times] \tag{3}$$

$$\boldsymbol{u}(t) = I^{-1}(\boldsymbol{T} - \dot{\boldsymbol{h}}) \tag{4}$$

then Eq. (2) can be written in the form

$$\dot{\omega} = F(\omega)\omega + \boldsymbol{u}(t) \tag{5}$$

As was shown in Ref. 3, there are eight primary models and infinite linear combinations of them that express Eq. (1) in the form of Eq. (5).

Equation (5) describes the SC correct dynamics; however, we usually do not know the exact values of I, T, $\boldsymbol{h}$, and its derivative. Therefore, we do not know the exact relationship between $\dot{\omega}$ and these elements. We express our lack of knowledge by adding a stochastic process to the dynamics equation of Eq. (1). We assume that this stochastic process, $\boldsymbol{w}'(t)$, is a zero mean white noise process. The resulting model, which is used by the estimator, is

$$\dot{\omega} = F(\omega)\omega + \boldsymbol{u}(t) + \boldsymbol{w}'(t) \tag{6}$$

If we denote ω by $\boldsymbol{x}$, then Eq. (6) can be written as

$$\dot{\boldsymbol{x}} = F(\boldsymbol{x})\boldsymbol{x} + \boldsymbol{u}(t) + \boldsymbol{w}'(t) \tag{7}$$

where obviously

$$F(\boldsymbol{x}) = I^{-1}[(I\boldsymbol{x} + \boldsymbol{h})\times] \tag{8}$$

For the time being we assume that we measure the angular rate, that is, $\boldsymbol{x}$; therefore, the measurement equation is

$$z_k = H\boldsymbol{x}_k + \boldsymbol{v}_k \tag{9}$$

where

$$H = I_3 \tag{10}$$

where $\boldsymbol{v}_k$ is a zero mean white measurement noise and I_3 is the third-dimensional identity matrix.

III. Angular-Rate Estimation

As mentioned in the introduction section, we use two filtering algorithms to estimate the angular rate. These algorithms are described next.

The dynamics equation presented in Eq. (7) is a nonlinear differential equation due to the term $F(\boldsymbol{x})\boldsymbol{x}$. A standard filter for this case is the extended KF (EKF). One can also apply the extended interlaced KF,[4] where three linear KFs are run in parallel. Other possibilities that are applicable to the form of nonlinearity presented in Eq. (7) are the pseudolinear Kalman (PSELIKA) filter[3] and the state-dependent algebraic Riccati equation (SDARE) filter, which were used successfully in Ref. 2. In view of their performance, the latter two filters are also used in this work.

A. PSELIKA Filter[3]

The PSELIKA filter algorithm disregards the nonlinearity and treats the dynamics system as if it were just a time-varying linear system; consequently, the ordinary KF algorithm is applied. First, the continuous differential equation (7) expressing the SC dynamics is discretized, and then the KF algorithm is applied as follows. First evaluate

$$W_k' = E\left\{\boldsymbol{w}'(t_k)\boldsymbol{w}'(t_k)^T\right\} \tag{11a}$$

$$R_k = E\left\{\boldsymbol{v}_k\boldsymbol{v}_k^T\right\} \tag{11b}$$

and choose an approximate value for the initial estimate of the rate vector. In the absence of such initial estimate, choose $\hat{\boldsymbol{x}}_0$ (which is $\hat{\omega}_0$) to be zero. Next, determine P_0, the initial covariance matrix of the estimation error, according to the confidence in the choice of $\hat{\boldsymbol{x}}_0$. The recurrence algorithm is then as follows.

1. Time Propagation

Let A_k be the discrete dynamics matrix obtained when $F(\boldsymbol{x})$ of Eq. (8) is discretized, and let $\boldsymbol{u}_k$ be the discrete deterministic input signal. Then propagate the state estimate according to

$$\hat{\boldsymbol{x}}_{k+1/k} = A_k\hat{\boldsymbol{x}}_{k/k} + \boldsymbol{u}_k \tag{12a}$$

and the covariance matrix according to

$$P_{k+1/k} = A_k P_{k/k} A_k^T + W_k' \tag{12b}$$

2. Measurement Update

Compute the Kalman gain as follows:

$$K_{k+1} = P_{k+1/k}H^T\left[HP_{k+1/k}H^T + R_{k+1}\right]^{-1} \tag{12c}$$

Update the estimate according to

$$\hat{\boldsymbol{x}}_{k+1/k+1} = \hat{\boldsymbol{x}}_{k+1/k} + K_{k+1}[z_{k+1} - H\hat{\boldsymbol{x}}_{k+1/k}] \tag{12d}$$

and update the covariance matrix using

$$P_{k+1/k+1} = [I_3 - K_{k+1}H]P_{k+1/k}[I_3 - K_{k+1}H]^T + K_{k+1}R_{k+1}K_{k+1}^T \tag{12e}$$

B. SDARE

The continuous-discrete-time SDARE filter that was used in Ref. 3 was based on the work of Cloutier et al.[11,12] Pappano and Friedland,[13] and Mracek et al.[14] That continuous-discrete-time filter for the continuous-time dynamics and the discrete-time measurement is as follows (see Ref. 3).

As with the PSELIKA filter, choose an approximate value for the initial estimate of the rate vector. In the absence of such initial estimate, choose again $\hat{\boldsymbol{x}}_0 = 0$.

1. Time Propagation

Propagate the state estimate according to

$$\hat{\boldsymbol{x}}_{k+1/k} = A_k\hat{\boldsymbol{x}}_{k/k} + \boldsymbol{u}_k \tag{13}$$

2. Measurement Update

At the measurement update time t_{k+1}, solve the following algebraic Riccati equation for P_{k+1}:

$$A(\hat{\boldsymbol{x}}_{k+1/k})P_{k+1} + P_{k+1}A^T(\hat{\boldsymbol{x}}_{k+1/k}) - P_{k+1}H^T R_{k+1}^{-1} H P_{k+1} + W_{k+1}' = 0 \tag{14a}$$

and compute the gain matrix

$$K_{k+1} = P_{k+1}H^T R_{k+1}^{-1} \tag{14b}$$

Finally, compute the updated state estimate:

$$\hat{\boldsymbol{x}}_{k+1/k+1} = \hat{\boldsymbol{x}}_{k+1/k} + K_{k+1}[z_{k+1} - H\hat{\boldsymbol{x}}_{k+1/k}] \tag{14c}$$

IV. Filtered Quaternion-Rate Approach

As mentioned before, it is possible to derive ω_r, a crude estimate of ω, using the first time derivative of the quaternion[2,3]; however, the resultant estimate is noisy. If ω_r is passed through a passive low-pass filter the noise may be filtered out at the expense of a delay.[2] Here we investigate the quality of the filtered rates when the two active filters described before are used to filter ω_r. First we show how ω_r is derived from $\dot{q}$, the differentiated quaternion. As is well known (e.g., see Ref. 10), the quaternion dynamics equation is

$$\dot{q} = \tfrac{1}{2}\Omega q \tag{15}$$

where

$$\Omega = \begin{bmatrix} 0 & \omega_z & -\omega_y & \omega_x \\ -\omega_z & 0 & \omega_x & \omega_y \\ \omega_y & -\omega_x & 0 & \omega_z \\ -\omega_x & -\omega_y & -\omega_z & 0 \end{bmatrix} \tag{16}$$

It is also known (e.g., see Ref. 2) that Eq. (15) can be written as

$$\dot{q} = \tfrac{1}{2}Q\omega \tag{17}$$

where

$$Q = \begin{bmatrix} q_4 & -q_3 & q_2 \\ q_3 & q_4 & -q_1 \\ -q_2 & q_1 & q_4 \\ -q_1 & -q_2 & -q_3 \end{bmatrix} \tag{18}$$

Note that

$$Q^T Q = I_3 \tag{19}$$

where I_3 is the third-dimensional identity matrix. When both sides of Eq. (17) are multiplied by $2Q^T$ and Eq. (19) is used, a rough estimate of the rate vector can be computed as follows:

$$\omega_r = 2Q^T\dot{q} \tag{20}$$

The dynamics equation for the estimator was introduced in Sec. II [see Eq. (7)]. In view of Eq. (20), the measurement equation [see Eq. (9)], which corresponds to the filter dynamics model, is

$$\omega_r = H_\omega \omega + v_\omega \tag{21a}$$

where

$$H_\omega = I_3 \tag{21b}$$

and v_ω is a zero-mean white noise. This defines the quaternion-rate approach.

The PSELIKA and the SDARE filters were used to obtain the angular rate from quaternion observations using the quaternion-rate approach. The data that were used to test this approach were real measurements downloaded from the Rossi X-Ray Timing Explorer (RXTE) satellite, which was launched on 30 December 1995. We chose a segment of data starting 4 January 1996 at 2130 and 1.148 s. The reference quaternion was based on the SC attitude as determined by its star trackers. Figure 1 presents ω, the nominal angular-rate, which was measured by the onboard gyros whose drift rate was of the order of 10^{-4} deg/h. Figure 2 presents the error between ω_r, the raw angular rate, and ω, the nominal rate. To quantify the error, a single figure of merit (FM) is computed. First, the average square error of each component is computed as follows:

$$\overline{e_i^2} = \frac{1}{T - t_0}\int_{t_0}^{T} e_i^2 \, dt, \qquad i = x, y, z$$

where t_0 is a time near the beginning of the data and T is the time at the end. This computation yields $\overline{e_x^2}$, $\overline{e_y^2}$, and $\overline{e_z^2}$. Then the FM is computed as $\mathrm{FM} = \sqrt{(\overline{e_x^2} + \overline{e_y^2} + \overline{e_z^2})}$. To exclude the transients, we set $t_0 = 100$ s. It was found that $\mathrm{FM}(2) = 7.3998 \times 10^{-3}$ deg/s, where FM(2) is the FM of Fig. 2. Figure 3 shows the estimation error when the PSELIKA filter was applied to ω_r. It was found that $\mathrm{FM}(3) = 1.5311 \times 10^{-3}$ deg/s. Finally, Fig. 4 shows the same when the SDARE filter was used, and it was found that $\mathrm{FM}(4) = 1.4550 \times 10^{-3}$ deg/s. As indicated by FM(2), the computed angular rate ω_r, particularly its x component, was rather noisy. When either the PSELIKA or the SDARE filter were applied to ω_r, other than a few spikes, the resulting $\hat{\omega}$ was smoother. In this example there was no real difference between the performance of the two filters [see FM(3) and FM(4)]. As expected, the computation of ω_r using Eq. (20) produced a noisy estimate due to the differentiation of the measured quaternion, which was corrupted by measurement noise, and the application of the PSELIKA filter to this ω_r filtered out most of the noise. When the SDARE rather than the PSELIKA filter was applied to ω_r, the filtered estimate of the angular rate was visually identical. In other words, the effect of the application of the SDARE filter was practically identical to that of the PSELIKA filter.

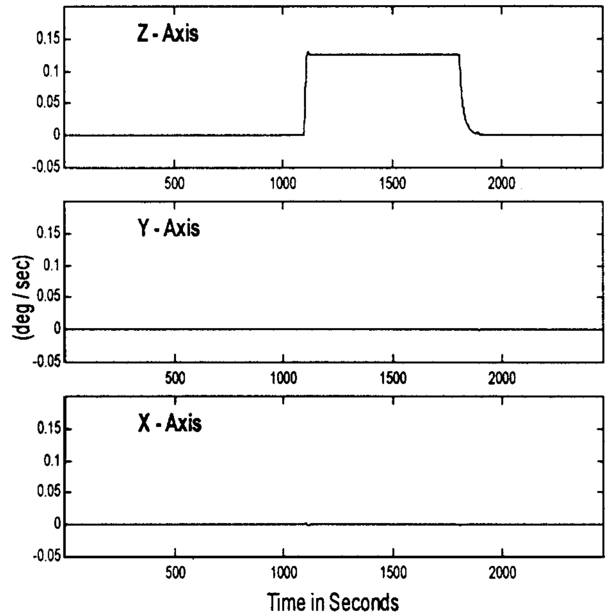

Fig. 1 Nominal angular rate.

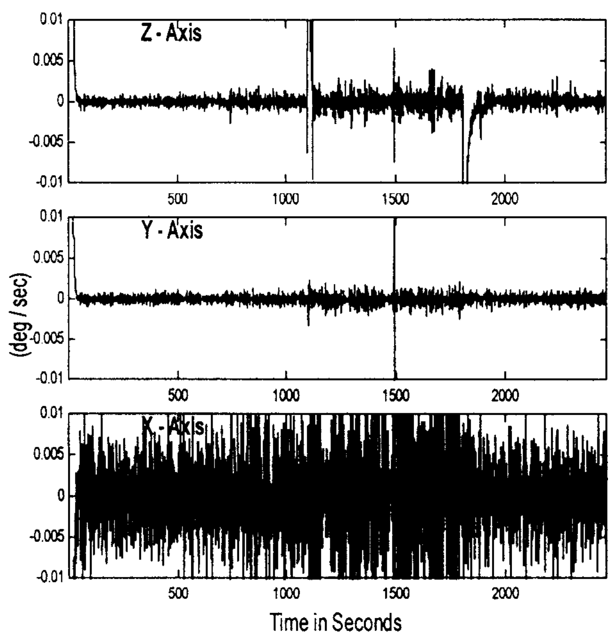

Fig. 2 Error between raw angular rate ω_r and nominal angular rate.

V. Quaternion Augmentation Approach

Although we also tested the quaternion-rate approach described in the preceding section, in this work we are mainly interested in estimating ω using the measured quaternion itself rather than its derivative. However, the quaternion is not a part of the state vector of the system [see Eqs. (6) and (7)]. One solution to this problem was examined in the preceding section. Another solution is the augmentation of the quaternion with the angular-rate state of Eqs. (6) and

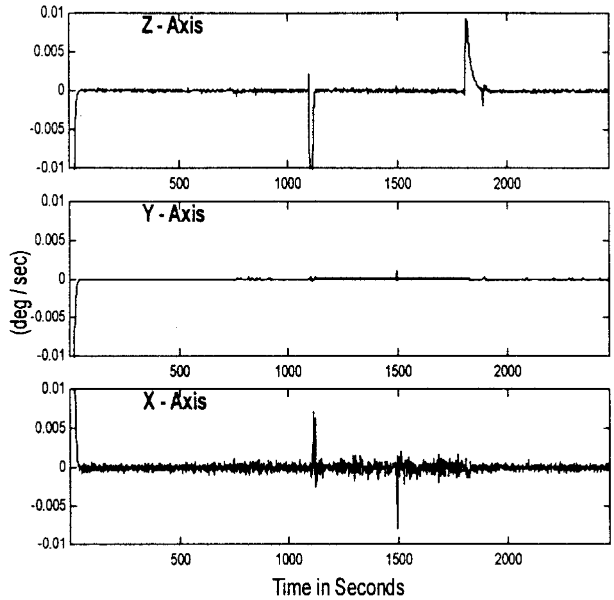

Fig. 3 Error in estimated angular rate $\hat{\omega}$ after applying PSELIKA to ω_r.

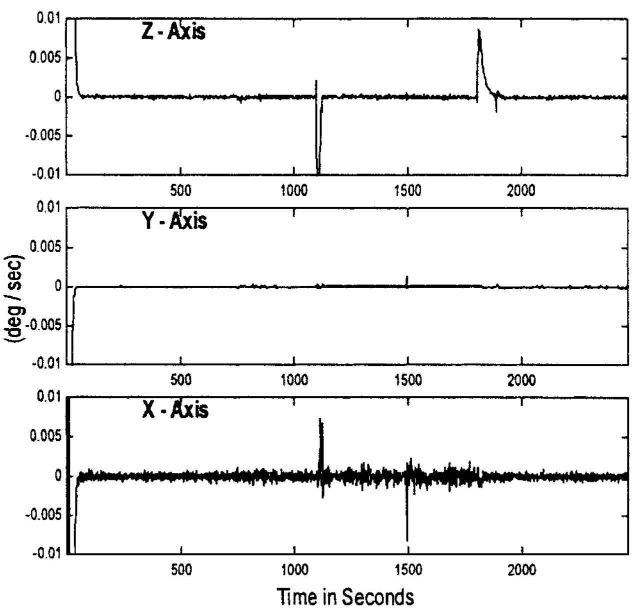

Fig. 4 Error in estimated angular rate $\hat{\omega}$ after applying SDARE to ω_r.

(7). For this, we can use the quaternion dynamics equation given in Eq. (15) and obtain the following model, which augments Eqs. (6) and (15):

$$\dot{y} = G'(y)y + e(t) + g(t) \tag{22}$$

where

$$y = \begin{bmatrix} \omega \\ q \end{bmatrix} \tag{23a}$$

$$G'y = \begin{bmatrix} F(\omega) & 0 \\ 0 & \frac{1}{2}\Omega \end{bmatrix} \tag{23b}$$

$$e(t) = \begin{bmatrix} u(t) \\ 0 \end{bmatrix} \tag{23c}$$

$$g(t) = \begin{bmatrix} w'(t) \\ 0 \end{bmatrix} \tag{23d}$$

The measurements of the quaternion are taken at discrete-time points; therefore, the measurement model is a discrete one. The discrete measurement model that corresponds to the dynamics model of Eq. (22) is

$$q_{m,k} = C\begin{bmatrix} \omega \\ q \end{bmatrix}_k + v_k \tag{24}$$

where $q_{m,k}$ is the measurement at time t_k,

$$C = \begin{bmatrix} 0 & 0 & 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 \end{bmatrix} \tag{25}$$

and v_k is the measurement noise at that time.

An inspection of the matrices $G'(y)$ of Eq. (23b) and C of the last equation reveals that even when ω is constant this pair is deterministically unobservable. This problem can be overcome though using the fact that Eq. (15) can be written as Eq. (17), which can also be written as

$$\dot{q} = \left[\tfrac{1}{2}Q \,|\, 0\right]\begin{bmatrix} \omega \\ q \end{bmatrix} \tag{26}$$

Therefore, Eq. (22) can be transformed into

$$\dot{y} = G(y)y + e(t) + g(t) \tag{27a}$$

where

$$G(y) = \begin{bmatrix} F(\omega) & 0 \\ \frac{1}{2}Q & 0 \end{bmatrix} \tag{27b}$$

We note that the measurement equation [see Eq. (24)] is unchanged, although the dynamics matrix of the system changes from $G'(y)$ to $G(y)$. Unlike the pair $G'(y)$ and C, the pair $G(y)$ and C is not necessarily deterministically unobservable. In fact, the results from the PSELIKA filter with the preceding model, which are presented in Fig. 5, show that the pair is observable even when ω is time varying. Moreover, in the computation of Ω that is needed in Eq. (23b), we use our best estimate of ω. At least initially, this estimate may be way off, yielding a wrong Ω and, consequently, a wrong $G'(y)$. On the other hand, in the computation of $G(y)$ given in Eq. (26b), we use Q rather than Ω. Because Q is based on the measured q, which is fairly accurate, we obtain an accurate $G(y)$.

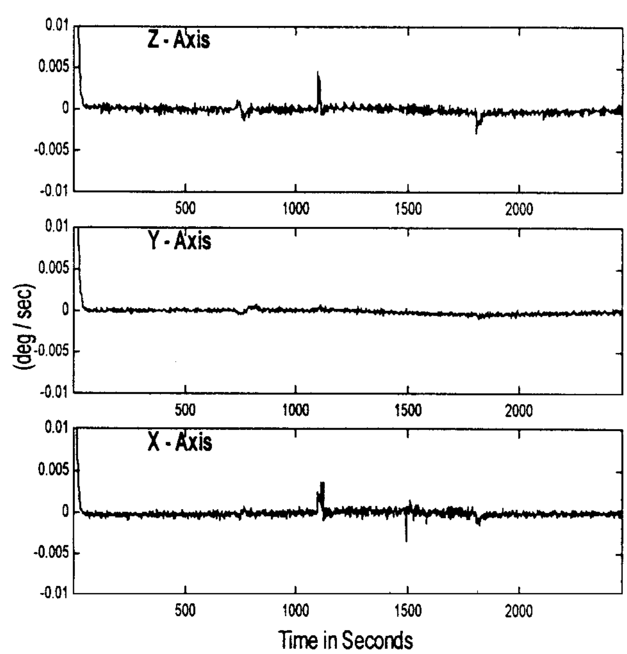

Fig. 5 Angular-rate estimation error after applying PSELIKA to the augmented model.

In other words, not only is the pair $\{G(y), C\}$ observable, the use of $G(y)$ yields a more accurate filter model than does $G'(y)$.

Whereas v_k, the measurement noise vector, can be assumed to be statistically independent over time, its components are correlated with one another; moreover, it cannot be assumed that v_k has a constantly zero mean. Consequently, we model the measurement noise as

$$v_k = v_{1,k} + v_{2,k} \tag{28}$$

where between the measurement points, $k-1$, k, and $k+1$, the noise component v_1 changes according to

$$\dot{v}_1 = -Nv_1 + \mu_1 \tag{29}$$

It is further assumed that $v_{2,k}$ is a zero-mean white noise process whose covariance matrix contains, in general, nonzero off-diagonal elements. As usual, the covariance matrix of the white noise vector μ_1, which drives $v_{1,k+1}$, is selected[15] to fit the covariance matrix of $v_{1,k+1}$. That matrix, too, may have nonzero off-diagonal elements to generate the correct covariance between the components of $v_{1,k+1}$.

Because the measurement noise has a nonwhite component, one needs to augment the nonwhite state with the existing state vector to form a new augmented state. The resultant model is then as follows:

$$\dot{x} = Fx + f + w \tag{30}$$

where

$$x = \begin{bmatrix} \omega \\ q \\ v_1 \end{bmatrix} \tag{31a}$$

$$F = \begin{bmatrix} F(\omega) & 0 & 0 \\ \frac{1}{2}Q & 0 & 0 \\ 0 & 0 & -N \end{bmatrix} \tag{31b}$$

$$f = \begin{bmatrix} u \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix} \tag{31c}$$

$$w = \begin{bmatrix} w' \\ \mathbf{0} \\ \mu_1 \end{bmatrix} \tag{31d}$$

Since

$$q_{m,k} = q_k + v_{1,k} + v_{2,k} \tag{32a}$$

then the corresponding discrete measurement equation is

$$z_{k+1} = Hx_{k+1} + v_{2,k+1} \tag{32b}$$

where

$$H = \begin{bmatrix} 0&0&0&1&0&0&0&1&0&0&0 \\ 0&0&0&0&1&0&0&0&1&0&0 \\ 0&0&0&0&0&1&0&0&0&1&0 \\ 0&0&0&0&0&0&1&0&0&0&1 \end{bmatrix} \tag{32c}$$

In the preceding analysis, we showed how to treat nonwhite measurement noise; however, in the data that were used here, the measurement noise was white; therefore, the model of Eqs. (27) was used in the filter. The results of estimating the angular rate when applying the PSELIKA filter to the augmented model are shown in Fig. 5. The FM of these results was found to be $\text{FM}(5) = 6.1839 \times 10^{-4}$ deg/s. When comparing Fig. 5 to Figs. 3 and 4, it is realized that the addition of q to the state vector yields a better filter. Note that the level of the spikes present in Figs. 3 and 4 was reduced when this filter was used.

VI. Simplified Filter Model

The dynamics models that were used in the preceding section can be drastically simplified by exchanging the SC nonlinear dynamics model with a simple first-order Markov model. This approach, which is common practice in target tracking, was applied recently to attitude determination.[8] The simplified filter dynamics equation takes the form

$$F_s = \begin{bmatrix} -T^{-1} & 0 & 0 \\ \frac{1}{2}Q & 0 & 0 \\ 0 & 0 & -N \end{bmatrix} \tag{33a}$$

The dynamics model is then

$$\dot{x}_s = F_s x_s + f + w_s \tag{33b}$$

where

$$x_s^T = \left[\omega_s^T \middle| q_s^T \middle| v_1^T\right] \tag{33c}$$

and f is as before and

$$w_s^T = \left[w_s'^T | \mathbf{0}^T | \mu_1^T\right] \tag{33d}$$

The covariance matrix of w_s has to be computed[15] and tuned. When the quaternion measurements are used to update the filter every second, there is almost no visible difference between the use of the elaborate rotational dynamics model and the simplified Markov model. However, if the updates occur at longer intervals, there is a remarkable difference between the two cases. Figure 6 presents the angular-rate estimation error when the elaborate angular dynamics is used and the PSELIKA filter, which is used to estimate the rates, is updated every 30 s. The FM computation of the error presented in Fig. 6 results in $\text{FM}(6) = 1.7975 \times 10^{-3}$ deg/s. When the elaborate model is replaced by the Markov model, the error in the resulting estimated rate is unacceptable. This is seen in Fig. 7, where the angular-rate estimation errors for this case are shown. Even before the start of the SC maneuver there is a marked difference between the results presented in Figs. 6 and 7; however, the remarkable difference between the two cases occur when the SC start the maneuver about its z axis. This is because the simple Markov model is incapable of capturing the SC maneuver. Note that in the computation of FM(6) and FM(7), we set $t_0 = 200$ s. Again, this was done to avoid the transients.

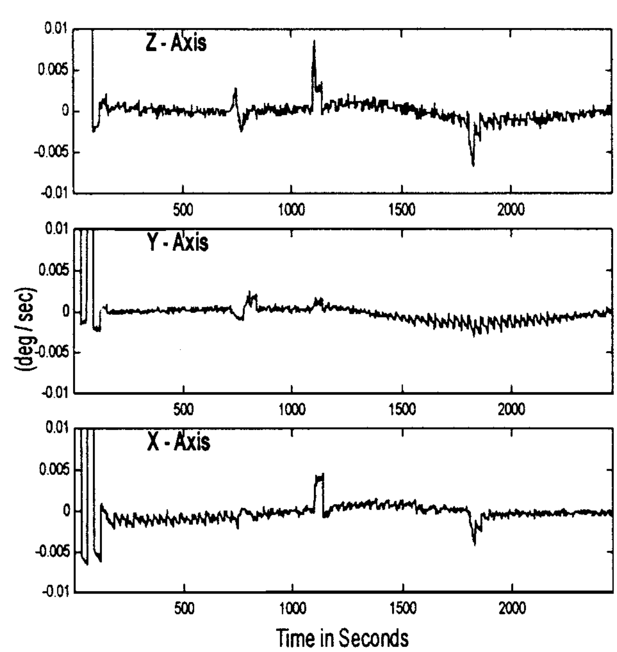

Fig. 6 Angular-rate estimation error for sparse measurements.

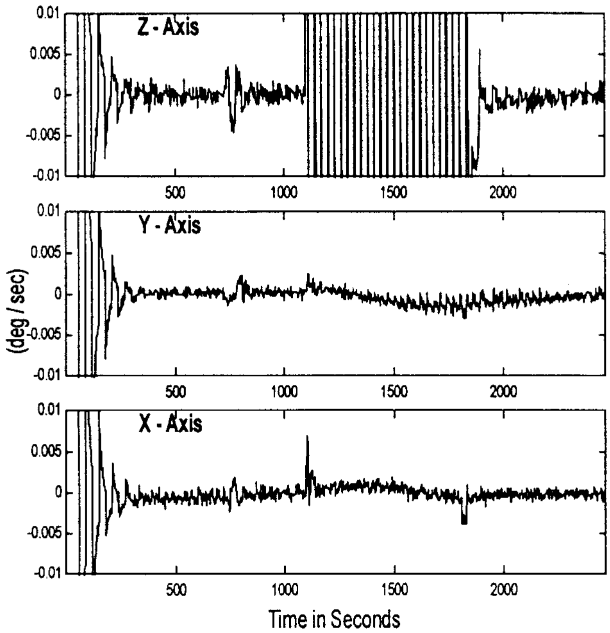

Fig. 7 Angular-rate estimation error with a simplified model and sparse measurements.

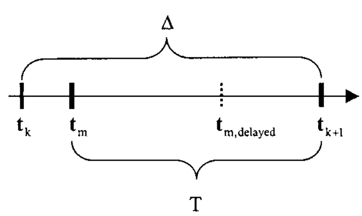

Fig. 8 Relative location of the measurement and its delay along the time line.

VII. Delayed Quaternion Measurements

The device that yields the quaternion measurements computes the quaternion after a star search; therefore, the quaternion is obtained with a time delay. Figure 8 presents the time points of the events related to the problem. The filter has to supply the best estimate of the angular rate to the SC attitude control system (ACS) at points t_k, t_{k+1}, etc. The measurements are of quaternions that express the attitude at different times though. Moreover, the measurements are delayed. This is shown in Fig. 8, where $t_{m,\text{delayed}}$ is the time where a measurement is obtained, but due to the delay it yields the attitude that existed at time t_m. There are several ways to process the delayed measurement to obtain an improved estimate at t_{k+1}. In the ensuing sections, we present two algorithms. According to the first algorithm, which we name updating before propagating, we perform a measurement update of the filter at time point $t_{m,\text{delayed}}$, and then propagate the outcome to time point t_{k+1}, where the information is passed on to the ACS.

It is also possible to first propagate the state estimate and covariance matrix (when PSELIKA is used) from time point t_k to the time point t_{k+1}, project the measurement from time point t_m to this time point, and only then perform a measurement update. We name this second algorithm updating after propagating. Both algorithms are further explained in the following sections.

A. Updating Before Propagating

The sequence of events concerning the propagation and the updating of the state estimate when using this algorithm is presented in Fig. 9. Because τ is known, it is possible to propagate the estimated state vector and the covariance matrix from time t_k to $t_{m,\text{delayed}}$, and because D is known, it is also possible to propagate the measurement. A measurement update is then performed at $t_{m,\text{delayed}}$, to yield $\hat{\boldsymbol{x}}_{m,\text{delayed}}(+)$, which is then propagated to time t_{k+1}, where it becomes $\hat{\boldsymbol{x}}_{k+1}(-)$. The propagation of the state vector from t_k to

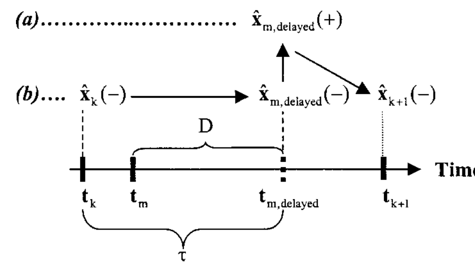

Fig. 9 Evolution of the state estimate over one time cycle: a) updated vectors and b) propagated vectors.

$t_{m,\text{delayed}}$ is done using the following discretized version of Eqs. (30) and (31):

$$\begin{bmatrix} \hat{\boldsymbol{\omega}}_{m,\text{delayed}}(-) \\ \hat{\boldsymbol{q}}_{m,\text{delayed}}(-) \\ \hat{\boldsymbol{\nu}}_{1,m,\text{delayed}}(-) \end{bmatrix} = \begin{bmatrix} e^{F(\hat{\omega})\tau} & 0 & 0 \\ \frac{1}{2}QF^{-1}(\hat{\omega})[e^{F(\hat{\omega})\tau} - I] & I & 0 \\ 0 & 0 & e^{-N\tau} \end{bmatrix} \times \begin{bmatrix} \hat{\boldsymbol{\omega}}_k(-) \\ \hat{\boldsymbol{q}}_k(-) \\ \hat{\boldsymbol{\nu}}_{1,k}(-) \end{bmatrix} + \begin{bmatrix} \boldsymbol{u}_k \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{34a}$$

and, when using PSELIKA, the covariance matrix is propagated using Eq. (12b) noting that

$$A_k = \begin{bmatrix} e^{F(\hat{\boldsymbol{\omega}})\tau} & 0 & 0 \\ \frac{1}{2}QF^{-1}(\hat{\omega})[e^{F(\hat{\boldsymbol{\omega}})\tau} - I] & I & 0 \\ 0 & 0 & e^{-N\tau} \end{bmatrix} \tag{34b}$$

[One need not worry about a possible singularity of $F(\hat{\omega})$ that may appear in the {2, 1} element of the discretized dynamics matrix in Eqs. (34) because $F^{-1}(\hat{\omega})$ is included in this element only to enable the expression of the term in a closed form. When this term is expressed in a power series form, the inverse of $F(\hat{\omega})$ is canceled out].

The propagation of the measured quaternion requires the propagation of the whole state vector by the interval D from time t_m to time $t_{m,\text{delayed}}$. For that, similarly to Eq. (34a), we use

$$\begin{bmatrix} \hat{\boldsymbol{\omega}}_{m,\text{delayed}} \\ \boldsymbol{q}^m_{m,\text{delayed}} \\ \hat{\boldsymbol{\nu}}_{1,m,\text{delayed}} \end{bmatrix} = \begin{bmatrix} e^{F(\hat{\omega})D} & 0 & 0 \\ \frac{1}{2}QF^{-1}(\hat{\omega})[e^{F(\hat{\omega})D} - I] & I & 0 \\ 0 & 0 & e^{-ND} \end{bmatrix} \times \begin{bmatrix} \hat{\boldsymbol{\omega}}_m \\ \boldsymbol{q}^m_m \\ \hat{\boldsymbol{\nu}}_{1,m} \end{bmatrix} + \begin{bmatrix} \boldsymbol{u}_m \\ \boldsymbol{0} \\ \boldsymbol{0} \end{bmatrix} \tag{35}$$

where the subscripts m and m,delayed denote values at times t_m and $t_{m,\text{delayed}}$, respectively, $\boldsymbol{q}^m_m$ is the measurement, and $\boldsymbol{q}^m_{m,\text{delayed}}$ is the propagated measurement needed for the update at $t_{m,\text{delayed}}$. Note that $\hat{\boldsymbol{\omega}}_m$ and $\hat{\boldsymbol{\nu}}_{1,m}$, needed in Eq. (35), are obtained by propagating $\hat{\boldsymbol{\omega}}_k(-)$ and $\hat{\boldsymbol{\nu}}_{1,k}(-)$, respectively, over the time span $\tau - D$ using the respective transition matrices $e^{F(\hat{\boldsymbol{\omega}})(\tau - D)}$ and $e^{-N(\tau - D)}$. Equation (32a) gives rise to the following expression for the propagated measurement:

$$\boldsymbol{q}^m_{m,\text{delayed}} = \boldsymbol{q}_{m,\text{delayed}} + \boldsymbol{\nu}_{1,m,\text{delayed}} + \boldsymbol{\nu}_{2,m,\text{delayed}} \tag{36}$$

Therefore, H, the measurement matrix, for the update is still that given in Eq. (32c). As for the computation of the covariance of the white measurement noise term, $\boldsymbol{\nu}_{2,m,\text{delayed}}$, it is realized that, although the equation $\dot{\boldsymbol{q}} = \frac{1}{2}Q\boldsymbol{\omega}$ was actually used to compute $\boldsymbol{q}^m_{m,\text{delayed}}$ [see (34a)], the result is identical to that obtained using the form $\dot{\boldsymbol{q}} = \frac{1}{2}\Omega\boldsymbol{q}$. When the latter equation is used, then the solution is

$$\boldsymbol{q}^m_{m,\text{delayed}} = \Phi(t_{m,\text{delayed}}, t_m)\boldsymbol{q}^m_m \tag{37}$$

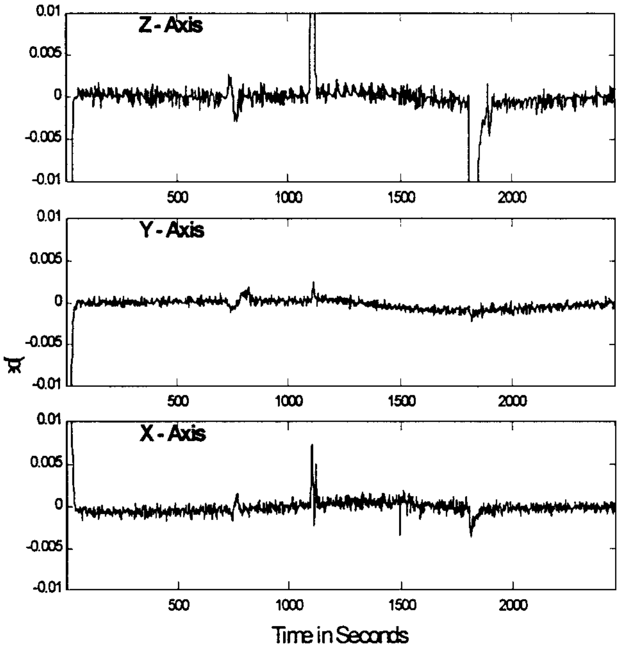

Fig. 10 Angular-rate estimation error for delayed measurements when updating before propagating.

where $\Phi(t_{m,\text{delayed}}, t_m)$ is a transition matrix that can be easily computed using

$$\Phi_q(\Delta) = \prod_{i=0}^{i=N-1} \exp\left[\frac{1}{2}\Omega(t_{i\delta}) \cdot \delta\right] \tag{38}$$

where δ is the length of a subinterval of D and $t_{i\delta}$ is the beginning of the ith such subinterval. Next $R_{2,m,\text{delayed}}$, the covariance matrix of the measurement noise for this update, has to be computed. From Eqs. (36) and (37), it is clear that

$$\boldsymbol{v}_{2,m,\text{delayed}} = \Phi(t_{m,\text{delayed}}, t_m)\boldsymbol{v}_{2,m} \tag{39}$$

Therefore, noting that $E\{\boldsymbol{v}_{2,m}\} = 0$, it is obvious that

$$R_{2,m,\text{delayed}} = \Phi(t_{m,\text{delayed}}, t_m)R_{2,m}\Phi^T(t_{m,\text{delayed}}, t_m) \tag{40}$$

where $R_{2,m}$ is the given covariance matrix of $\boldsymbol{v}_{2,m}$, the measurement white noise at time t_m [see Eq. (32a)]. Once the measurement update is completed, the outcome, $\hat{\boldsymbol{x}}_{m,\text{delayed}}(+)$, is propagated to time point t_{k+1} over the time span $\Delta - \tau$ exactly as the state estimate was propagated from t_k to $t_{m,\text{delayed}}$ over the time span τ. The propagated state estimate becomes $\hat{\boldsymbol{x}}_{k+1}(-)$. Note that when the SDARE filter is used, there is no covariance matrix propagation. Figure 10 presents the estimation error when the PSELIKA filter was used to estimate the angular velocity from delayed measurement and the updating before propagating approach was used. In this case, $\Delta = T = 1$ s, and $D = \tau = 0.5$ s. Note that in this example t_m and t_k coincide. It was found that FM(10) $= 5.842090 \times 10^{-3}$ deg/s. A comparison of FM(10) to FM(5) reveals that the delay introduced an error that is an order of magnitude larger than that obtained when the measurements had no delay.

B. Updating After Propagating

The sequence of events concerning the propagation and the updating of the state estimate when using this algorithm is presented in Fig. 11. Here first the state estimate is propagated by Δ to time point t_{k+1}, and if PSELIKA is used, the covariance matrix is also propagated.

Then the measurement $\boldsymbol{q}_m$ is also propagated by T to t_{k+1}, and an update is performed there yielding $\hat{\boldsymbol{x}}_{k+1}(+)$. The propagation of the state estimate and the measurement are performed as they were before using the appropriate time intervals. Figure 12 presents the estimation error when the PSELIKA filter was used to estimate the angular velocity from delayed measurement and the updating after propagating approach was used. Also in this run $\Delta = T = 1$ s. The computed FM for this case was computed as FM(12) $= 8.412262 \times 10^{-4}$ deg/s. A comparison between FM(12) and FM(10) shows that a more accurate estimate is obtained when the update is done after propagating the state and the measurement to the time point t_{k+1}, where the next estimate has to be fed into the attitude control system.

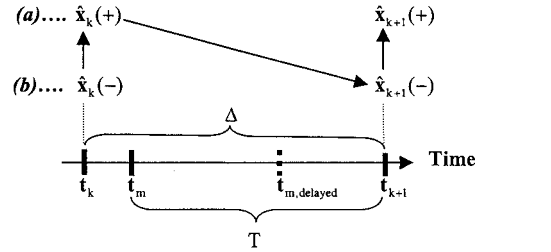

Fig. 11 Evolution of the state estimate over one time cycle: a) updated vectors and b) propagated vectors.

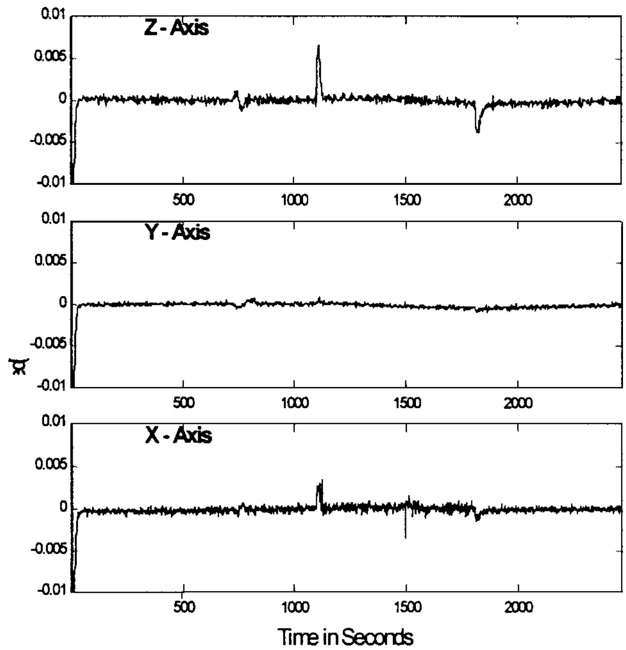

Fig. 12 Angular-rate estimation error for delayed measurements when updating after propagating.

VIII. Conclusions

We examined algorithms for estimating the angular-rate vector of satellites using quaternion measurements without differentiation. The notion examined in this work is based on the ability to obtain quaternion measurements directly from a cluster of star trackers. For comparison, we also examined the approach of extracting the angular rate from quaternion differentiation. Both approaches utilize a Kalman filter. Two filters were examined. One was the PSELIKA filter, and the other was a special KF that was based on the use of the solution of SDARE to compute the Kalman gain matrix and, thus, eliminate the need to propagate and update the filter covariance matrix. The two filters relied on the ability to decompose the nonlinear rate-dependent part of the rotational dynamics equation of a rigid body into a product of an angular-rate-dependent matrix and the angular-rate vector itself. This nonunique decomposition enabled the treatment of the nonlinear SC dynamics model as a linear one and, consequently, the application of the PSELIKA filter. It also enabled the application of the SDARE filter.

When using the quaternion measurements to obtain angular rate without differentiation, the kinematics equation of the quaternion has to be incorporated into the filter dynamics model. This can be done in two ways. It was shown that only one way can be used because only this way yields an observable system.

Real spacecraft data were used to test the suggested algorithms. As expected, when rate determination was based on quaternion differentiation, the resulting angular-rate was noisy. When either one of the filters was used, the noise was suppressed without causing delays in the estimated angular-rate components.

The replacement of the elaborate rotational dynamics by a simple first-order Markov model was also examined. It was found that, although the use of such a simple model was sufficient when frequent measurement updates were possible, it was totally inadequate when only sparse quaternion measurements were available.

A device that yields the quaternion measurements computes the quaternion after a star search; therefore the quaternion is obtained with a time delay. However, the filter has to supply the best estimate of the angular rate to the SC attitude control system on time. In this work, two algorithms were presented to overcome the delay problem. According to the first algorithm, *updating before propagating*, a measurement update of the filter is performed at the time when the measurement is obtained and then propagated to the time point where the information is passed on to the ACS. In the second algorithm the state estimate, the covariance matrix (when PSELIKA is used), and the measurement are first propagated to the time point where the angular rate has to be passed on to the ACS, and only then a measurement update is performed. This second algorithm was named *updating after propagating*. Both algorithms were tested, and it was found in the cases tested that the updating after propagating algorithm yielded a more accurate estimate.

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